Holomorphic fiber bundle with Stein base and Stein fibers

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Summary. In this article, we prove that if $\Pi : X \to \Omega$ is a surjective holomorphic map, with Ω a Stein space and X a complex manifold of dimension $n \geq 3$, and if, for every $x \in \Omega$ there exists an open neighborhood U such that $\Pi^{-1}(U)$ is Stein, then X is Stein.

Key words: Stein spaces; Stein mophism; q-complete spaces; hyperconvex sets.

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1 Introduction

A surjective holomorphic map $\Pi : X \to Y$ between complex spaces is said to be a Stein morphism, if for every point $y \in Y$, there exists an open neighborhood U of y such that $\Pi^{-1}(U)$ is Stein.

In 1977, Skoda [12] showed that a locally trivial analytic fiber bundle $\Pi : X \to \Omega$ with Stein base and Stein fibers is not necessarily a Stein manifold. Thus giving a negative answer to a conjecture proposed in 1953 by J-P. Serre [11]. In the counterexample provided by Skoda the base Ω is an open set in $\mathbb{C}$ and the holomorphic functions on X are constant along the fibers $\Pi^{-1}(t)$, $t \in \Omega$. It follows (cf. J-P. Demailly [4]) that the cohomology group $H^1(X, \mathcal{O}_X)$ is an infinitely dimensional complex vector space.

At the same time, Fornaess [6] solved by means of a 2-dimensional counterexample the analogous problem for the Stein morphism.

There are still other counterexamples to the problem of Serre by J-P. Demailly [5] in 1978 and by Coeuré and Loeb [3] in 1985.

Note however that if $\Pi : X \to \Omega$ is a Stein morphism with a Stein base Ω, it follows from [8] that for any coherent analytic sheaf $\mathcal{F}$ on X the cohomology groups $H^p(X, \mathcal{F})$ vanish for all $p \geq 2$. So it is therefore natural to raise the question whether X is 2-complete. The main new result of this article is to give a positive answer to this problem when X is a complex manifold.

We obtain this result as consequence of the following theorem

theorem 1 *-Let X be a complex manifold of dimension $n \geq 3$, and Ω a complex space such that there exists a Stein morphism $\Pi : X \to \Omega$. If Ω is Stein, then X itself is Stein.*

Proof of theorem 1

We consider a covering $\mathcal{V} = (V_i)_{i\in\mathbb{N}}$ of Ω by open sets $V_i \subset \Omega$ such that $\Pi^{-1}(V_i)$ is Stein for all $i \in \mathbb{N}$. By the Stein covering lemma of Sthelé [13], there exits a locally finite covering $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ of Ω by Stein open subsets $U_i \subset\subset \Omega$ such that $\mathcal{U}$ is a refinement of $\mathcal{V}$ and $\bigcup_{i\leq j} U_i$ is Stein for all j. Moreover, there exists a continuous strictly psh function ϕ_{j+1} in $\bigcup_{i\leq j+1} U_i$ such that

$$\bigcup_{i\leq j} U_i \cap U_{j+1} = \{x \in U_{j+1} : \phi_{j+1}(x) < 0\}$$

Note also that $\Pi^{-1}(U_i)$ is Stein for all $i \in \mathbb{N}$ and, if $X_j = \Pi^{-1}(\bigcup_{i\leq j} U_i)$ and $X'_{j+1} = \Pi^{-1}(U_{j+1})$, then $X_j \cap X'_{j+1} = \{x \in X'_{j+1} : \phi_{j+1} o\Pi(x) < 0\}$ is Runge in X'_{j+1}.

lemma 1 *-Under the above assumptions, the sets X_j, $j \in \mathbb{N}$, are Stein*

Proof. The proof is by induction on j.

For $j = 0$, this is clear, since $\Pi^{-1}(U_i)$ is Stein for all $i \in \mathbb{N}$. Now let $j \geq 1$ and suppose that X_j is Stein. Let $Y_j = \{x \in X_j : \phi_{j+1} o\Pi(x) > 0\}$ and $Y'_{j+1} = \{x \in X'_{j+1} : \phi_{j+1} o\Pi(x) > 0\}$. Then $H^p(Y_j, \mathcal{O}_X) \cong H^p(Y'_{j+1}, \mathcal{O}_X) = 0$ for $1 \leq p \leq n-2$. In fact, let $\xi \in X'_{j+1}$ and $V \subset X'_{j+1}$ a hyperconvex open neighborhood of ξ. Let $\psi : V \to]-\infty, 0[$ be a continuous strictly plurisubharmonic function. Then $\psi_k = \frac{1}{k}\psi + \phi_{j+1} o\Pi$, $k \geq 1$, is an increasing

sequence of continuous strictly psh functions such that $\psi_k \to (\phi_{j+1} o \Pi)|_V$. Let $V_k = \{x \in V : \psi_k(x) > 0\}$, $k \geq 1$. Then for every point $\xi' \in V$ such that $\psi_k(\xi') = 0$ there exists, by [2, lemma 2], a fundamental system of connected Stein neighborhoods $U \subset\subset V$ of ξ' such that $H^r(U \cap V_k, \mathcal{O}_X) = 0$ for $1 \leq r \leq n-2$ and the restriction map $H^o(U, \mathcal{O}_X) \to H^o(U \cap V_k, \mathcal{O}_X)$ is an isomorphism for every $k \in \mathbb{N}$. Then a similar proof as in [1, lemma 2] shows that if $S'_k = \{x \in V : \psi_k(x) \leq 0\}$, then $H^p_{S'_k}(V, \mathcal{O}_X) = 0$ for $p \leq n-1$, where $H^p_{S'_k}(V, \mathcal{O}_X)$ is the p-th group of cohomology of V with support in S'_k. Hence the exact sequence of local cohomology

$$\cdots \to H^p_{S'_k}(V, \mathcal{O}_X) \to H^p(V, \mathcal{O}_X) \to H^p(V_k, \mathcal{O}_X) \to H^{p+1}_{S'_k}(V, \mathcal{O}_X) \to \cdots$$

shows that $H^p(V_k, \mathcal{O}_X) = 0$ for $1 \leq p \leq n-2$ and
$H^o(V_k, \mathcal{O}_X) \cong H^o(V, \mathcal{O}_X)$ for all $k \in \mathbb{N}$. Since $V \cap Y'_{j+1}$ is an increasing union of V_k, $k \in \mathbb{N}$, it follows from [2, lemma, p. 250] that
$H^p(V \cap Y'_{j+1}, \mathcal{O}_X) = 0$ for $1 \leq p \leq n-2$ and $H^o(V, \mathcal{O}_X) \to H^o(V \cap Y'_{j+1}, \mathcal{O}_X)$ is an isomorphism. Since every point has a fundamental system of hyperconvex neighborhoods, it follows from [7] that if
$S'_{j+1} = \{x \in X'_{j+1} : \phi_{j+1} o \Pi(x) \leq 0\}$, then $\underline{H^p_{S'_{j+1}}}(\mathcal{O}_X) = 0$ for $0 \leq p \leq n-1$, where $\underline{H^p_{S'_{j+1}}}(\mathcal{O}_X)$ is the cohomology sheaf of X'_{j+1} with coefficients in $\mathcal{O}_X$ and support in S'_{j+1}. Moreover, there is a spectral sequence

$$H^p_{S'_{j+1}}(X'_{j+1}, \mathcal{O}_X) \Longleftarrow E_2^{p,q} = H^p(X'_{j+1}, \underline{H^p_{S'_{j+1}}}(\mathcal{O}_X))$$

Since $\underline{H^p_{S'_{j+1}}}(\mathcal{O}_X) = 0$ for $p \leq n-1$, then $H^p_{S'_{j+1}}(X'_{j+1}, \mathcal{O}_X) = 0$ for $p \leq n-1$. Now by using the exact sequence of local cohomology

$$\cdots \to H^p_{S'_{j+1}}(X'_{j+1}, \mathcal{O}_X) \to H^p(X'_{j+1}, \mathcal{O}_X) \to H^p(Y'_{j+1}, \mathcal{O}_X) \to H^{p+1}_{S'_{j+1}}(X'_{j+1}, \mathcal{O}_X) \to \cdots$$

we see that $H^p(Y'_{j+1}, \mathcal{O}_X) \cong H^p(X'_{j+1}, \mathcal{O}_X) = 0$ for $1 \leq p \leq n-2$.
Similarly $H^p(Y_j, \mathcal{O}_X) = 0$ for $1 \leq p \leq n-2$.
We have $Y_{j+1} = \{x \in X_{j+1} : \phi_{j+1} o \Pi(x) > 0\} = Y_j \cup Y'_{j+1}$. Since $Y_j \cap Y'_{j+1} = \emptyset$, then $H^p(Y_{j+1}, \mathcal{O}_X) = 0$ for $1 \leq p \leq n-2$.
On the other hand, if $S_{j+1} = \{x \in X_{j+1} : \phi_{j+1} o \Pi(x) \leq 0\}$, then we can prove

exactly as for $H^p_{S'_{j+1}}(X'_{j+1},\mathcal{O}_X)$ that $H^p_{S_{j+1}}(X_{j+1},\mathcal{O}_X)=0$ for $p\leq n-1$. Now the exact sequence of local cohomology

$$\cdots \to H^1_{S_{j+1}}(X_{j+1},\mathcal{O}_X) \to H^1(X_{j+1},\mathcal{O}_X) \to H^1(Y_{j+1},\mathcal{O}_X)=0$$

implies that $H^1(X_{j+1},\mathcal{O}_X)=0$. This proves that X_{j+1} is Stein. (See [9]).

End of the proof of theorem 1

We consider for every $j\geq 1$ the long exact sequence of cohomology associated to the Mayer-Vietoris sequence

$$\begin{gathered}\cdots \to H^o(X_{j+1},\mathcal{O}_X) \to H^o(X_j,\mathcal{O}_X)\oplus H^o(X'_{j+1},\mathcal{O}_X) \to \\ H^o(X_j\cap X'_{j+1},\mathcal{O}_X) \to H^1(X_{j+1},\mathcal{O}_X) \to \cdots\end{gathered}$$

Since $H^1(X_{j+1},\mathcal{O}_X)=0$ and the restriction map

$$H^o(X'_{j+1},\mathcal{O}_X) \to H^o(X_j\cap X'_{j+1},\mathcal{O}_X)$$

has dense image, then, by ([2], proof of Proposition 19), one deduces that X_j is Runge in X_{j+1}, which implies (see [14]) that $X=\bigcup\limits_{j\geq 1} X_j$ is Stein.

Corollary 1 *-Let $\Pi : X \to \Omega$ be a Stein morphism with a Stein base Ω. Assume that*
a) X is a complex manifold of dimension $n\geq 3$
or
b) X a complex space of dimension 2.
Then X is 2*-complete.*

Proof

If $dim(X)=n\geq 3$, then, by theorem 1, X is Stein.
Suppose that $dim(X)=2$ and let A be a compact connected analytic subset of X. Since Ω is Stein, then $\Pi(A)$ is reduced to one point. This implies that A is contained in one fiber. Since the fibers of Π are Stein, then A is 0-dimensional, which proves by a theorem of Oshawa [10] that X is 2-complete.

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